Final Exam — Analysis (WPMA14004)

Monday 28 January 2019, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (6 + 6 + 3 = 15 points)

Consider the set $A = \left\{ \frac{1}{p} - \frac{1}{q} \, : \, p, q \in \mathbb{N} \right\}$.

(a) Prove that $\sup A = 1$.

(b) Prove that $\inf A = -1$.

(c) Does the set A contain all its limit points?

Problem 2 (5 + 5 + 5 = 15 points)

Decide whether each of the following series converges or diverges. Motivate your answers!

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{2^n + 3^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{p_n}$ where p_n is the n-th prime number (e.g. $p_1 = 2$ and $p_6 = 13$).

Problem 3 (15 points)

For sets $A, B \subseteq \mathbb{R}$ we define their sum as

$$A + B = \{ a + b \, : \, a \in A, b \in B \}.$$

Prove that if A and B are both compact, then $A + B$ is also compact.
(Hint: use the *definition* of compactness!)

Problem 4 (5 + 5 + 5 = 15 points)

Let $p \in \mathbb{R}$, and consider the following function:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^p \sin(1/x) & \text{if } x > 0. \end{cases}$$

(a) Show that if f is continuous at $x = 0$ then $p > 0$. (Hint: consider sequences $x_n \to 0$.)

(b) Conversely, show that if $p > 0$ then f is continuous at $x = 0$.

(c) Assume that $p = 1$. Is f differentiable at $x = 0$?

Problem 5 (3 + 4 + 4 + 4 = 15 points)

Consider the following sequence of functions:

$$f_n(x) = \frac{n^2 x}{1 + n^3 x^2}.$$

(a) Show that (f_n) converges pointwise to $f(x) = 0$ for all $x \in [0, \infty)$.

(b) Show that the function f_n has a maximum at $x_n = 1/n\sqrt{n}$.

(c) Does the sequence (f_n) converge uniformly to f on $[0, \infty)$?

(d) Does the sequence (f_n) converge uniformly to f on $[2, \infty)$?

Problem 6 (2 + 8 + 5 = 15 points)

Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ x - 1 & \text{if } 1 \leq x \leq 2. \end{cases}$$

(a) Sketch the graph of this function.

(b) Prove that f is integrable on $[0, 2]$. (Hint: consider $g(x) = f(x) - x$.)

(c) Let $F(x) = \int_0^x f$. Is F differentiable at $x = 1$? If so, what is $F'(1)$?

End of test (90 points)

Solution of Problem 1 (6 + 6 + 3 = 15 points)

(a) For $p \in \mathbb{N}$ we have $p \geq 1$ which implies that

$$\frac{1}{p} - \frac{1}{q} \leq 1 - \frac{1}{q} < 1$$

for all $p, q \in \mathbb{N}$. This shows that 1 is indeed an upper bound of the set A.

(2 points)

To show that 1 is the least upper bound we can follow two different strategies.

Strategy 1. Let u be an arbitrary upper bound for A:

$$\frac{1}{p} - \frac{1}{q} \leq u \quad \text{for all} \quad p, q \in \mathbb{N}.$$

In particular, this implies that

$$1 - \frac{1}{q} \leq u \quad \text{for all} \quad q \in \mathbb{N}.$$

The Order Limit Theorem implies that $1 \leq u$ when taking the limit $q \to \infty$. Therefore, 1 is the least upper bound of A.

(4 points)

Strategy 2. Let $\epsilon > 0$ be arbitrary. By the Archimedean Principle there exists $q \in \mathbb{N}$ such that $1/q < \epsilon$. This implies that $1 - 1/q > 1 - \epsilon$, which shows that $1 - \epsilon$ is *not* an upper bound for the set A. Therefore, 1 is the least upper bound of A.

(4 points)

(b) For $q \in \mathbb{N}$ we have $q \geq 1$ which implies that

$$\frac{1}{p} - \frac{1}{q} > -\frac{1}{q} \geq -1.$$

for all $p, q \in \mathbb{N}$. This shows that -1 is indeed a lower bound of the set A.

(2 points)

To show that 1 is the least upper bound we can follow two different strategies.

Strategy 1. Let ℓ be an arbitrary lower bound for A:

$$\ell \leq \frac{1}{p} - \frac{1}{q} \quad \text{for all} \quad p, q \in \mathbb{N}.$$

In particular, this implies that

$$\ell \leq \frac{1}{p} - 1 \quad \text{for all} \quad p \in \mathbb{N}.$$

The Order Limit Theorem implies that $\ell \leq -1$ when taking the limit $p \to \infty$. Therefore, -1 is the greatest lower bound of A.

(4 points)

Strategy 2. Let $\epsilon > 0$ be arbitrary. By the Archimedean Principle there exists $p \in \mathbb{N}$ such that $1/p < \epsilon$. This implies that $1/p - 1 < -1 + \epsilon$, which shows that $-1 + \epsilon$ is *not* a lower bound for the set A. Therefore, -1 is the greatest lower bound of A.

(4 points)

(c) Consider the sequence $x_n = 1 - 1/n$. Clearly, $x_n \neq 1$ and $x_n \in A$ for all $n \in \mathbb{N}$. In addition, $x_n \to 1$. This shows that $x = 1$ is a limit point of A.
(1 point)

However,

$$\frac{1}{p} - \frac{1}{q} \leq 1 - \frac{1}{q} < 1 \quad \text{for all} \quad p, q \in \mathbb{N},$$

which implies that $1 \notin A$. Therefore, A does not contain (all) its limit points.
(2 points)

Note. A similar reasoning shows that -1 is a limit point of A which is not contained in A.

Solution of Problem 2 (5 + 5 + 5 = 15 points)

(a) *Method 1.* Note that the sequence

$$a_n := \frac{6^n}{2^n + 3^n} = \frac{1}{(\frac{2}{6})^n + (\frac{3}{6})^n}$$

is unbounded since the denominator on the right-hand side converges to 0. A necessary condition for a series $\sum_{n=1}^{\infty} a_n$ to converge is that $a_n \to 0$. In this particular case, this necessary condition is not satisfied. Hence, the series diverges.
(5 points)

Method 2. Note that

$$a_n := \frac{6^n}{2^n + 3^n} > \frac{5^n}{2^n + 3^n} = \frac{(2+3)^n}{2^n + 3^n} > \frac{2^n + 3^n}{2^n + 3^n} = 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

A necessary condition for a series $\sum_{n=1}^{\infty} a_n$ to converge is that $a_n \to 0$. In this particular case, this necessary condition is not satisfied. Hence, the series diverges.
(5 points)

Method 3. Note that

$$a_n := \frac{6^n}{2^n + 3^n} > \frac{6^n}{3^n + 3^n} = \frac{6^n}{2 \cdot 3^n} = 2^{n-1} \quad \text{for all} \quad n \in \mathbb{N}.$$

The geometric series $\sum_{n=0}^{\infty} r^n$ diverges when $r > 1$. In particular, the series $\sum_{n=1}^{\infty} 2^{n-1}$ diverges. By the Comparison Test, the series $\sum_{n=1}^{\infty} a_n$ diverges.
(5 points)

(b) Note that

$$b_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2 + n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}.$$

Hence, the given series is of telescope type. The p-th partial sum of the series is given by

$$\sum_{n=1}^{p} b_n = 1 - \frac{1}{\sqrt{p+1}},$$

which obviously converges. This shows that the given series is convergent.
(5 points)

(c) Note that the prime numbers form a positive, increasing sequence: $0 < p_n < p_{n+1}$ for all $n \in \mathbb{N}$. Therefore, their reciprocals form a positive decreasing sequence:

$$0 < \frac{1}{p_{n+1}} < \frac{1}{p_n} \quad \text{for all} \quad n \in \mathbb{N}.$$

(2 points)

Also note that that the prime numbers form an unbounded sequence, which implies that $\lim 1/p_n = 0$. (Alternatively, one can argue that $1/p_n$ is a subsequence of $1/n$ and hence $\lim 1/p_n = \lim 1/n = 0$.)
(2 points)

By the Alternating Series Test the series $\sum_{n=1}^{\infty} (-1)^{n+1}/p_n$ converges.
(1 point)

Solution of Problem 3 (15 points)

Let (x_n) be an arbitrary sequence in the set $A+B$. Then there exists a sequence (a_n) in A and a sequence (b_n) in B such that $x_n = a_n + b_n$ for all $n \in \mathbb{N}$.
(3 points)

Since A is compact, the sequence (a_n) has a convergent subsequence (a_{n_k}) such that $a_{n_k} \to a$ with $a \in A$.
(4 points)

Note that (b_{n_k}) can be considered as a sequence in B in its own right. Since B is compact, the sequence (b_{n_k}) has a subsequence $(b_{n_{kj}})$ such that $b_{n_{kj}} \to b$ with $b \in B$.
(4 points)

By the Algebraic Limit Theorem it follows that $x_{n_{kj}}$ is a convergent sequence and that

$$x = \lim x_{n_{kj}} = \lim a_{n_{kj}} + \lim b_{n_{kj}} = a + b \in A + B.$$

This proves that the set $A+B$ is also compact since we have shown that every sequence in $A+B$ has a convergent subsequence of which the limit is again an element of $A+B$.
(4 points)

Solution of Problem 4 (5 + 5 + 5 = 15 points)

(a) If f is continuous at $x = 0$, then $f(x_n) \to f(0) = 0$ for all convergent sequences x_n with $x_n \to 0$.

Consider the following sequence:

$$x_n = \frac{2}{(4n-3)\pi},$$

which are the positive x-values for which $\sin(1/x) = 1$. Clearly, $x_n \to 0$. Therefore, if f is continuous at $x = 0$, then $f(x_n) \to 0$, or, equivalently,

$$\frac{1}{(4n-3)^p} \to 0.$$

This is the case if and only if $p > 0$.
(5 points)

(b) *Alternative 1.* Assume that $p > 0$. If x_n is any convergent sequence with $x_n \to 0$ then

$$|f(x_n) - f(0)| = |f(x_n)| \leq |x_n^p \sin(1/x_n)| \leq |x_n|^p \to 0,$$

where in the last step it has been used that the standard function $g(x) = x^p$ is continuous at $x = 0$ and $g(0) = 0$.

This proves that f is continuous at $x = 0$. (Note: if $x_n < 0$ for some n, then $f(x_n) = 0$. Therefore, we have used in inequality for $|f(x_n)|$ rather than an equality.)
(5 points)

Alternative 2. Assume that $p > 0$. Let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon^{1/p}$. If $|x - 0| < \delta$, then

$$|f(x) - f(0)| = |f(x)| \leq |x^p \sin(1/x)| \leq |x|^p = |x - 0|^p < \delta^p = \epsilon.$$

This proves that f is continuous at $x = 0$. (Note: if $x < 0$ for, then $f(x) = 0$. Therefore, we have used in inequality for $|f(x)|$ rather than an equality.)
(5 points)

(c) Assume that $p = 1$. The difference quotient of f is given by

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} 0 & \text{if } x < 0, \\ \sin(1/x) & \text{if } x > 0. \end{cases}$$

The limit of this difference quotient as $x \to 0$ does not exist. Indeed, consider the sequences

$$x_n = -\frac{1}{n} \quad \text{and} \quad y_n = \frac{2}{(4n-3)\pi}.$$

Then $f(x_n) = 0$ for all $n \in \mathbb{N}$ so that $(f(x_n) - f(0))/(x_n - 0) \to 0$, whereas

$$\frac{f(y_n) - f(0)}{y_n - 0} = \sin(1/y_n) = 1$$

for all $n \in \mathbb{N}$.
(5 points)

Solution of Problem 5 (3 + 4 + 4 + 4 = 15 points)

(a) Clearly, $f_n(0) = 0$ for all $n \in \mathbb{N}$ so $\lim f_n(0) = 0$. If $x > 0$, then

$$|f_n(x)| < \frac{1}{nx} \to 0 \quad \text{as} \quad n \to \infty.$$

(3 points)

(b) Differentation gives

$$f_n'(x) = \frac{(1+n^3x^2)n^2 - n^2x \cdot 2n^3x}{(1+n^3x^2)^2} = \frac{n^2(1-n^3x^2)}{(1+n^3x^2)^2}.$$

Clearly, $f_n'(x) = 0$ if and only if $x = \pm 1/n\sqrt{n}$.
(3 points)

At $x = 1/n\sqrt{n}$ the function f_n' changes sign from positive to negative. Hence, the function f_n attains a local maximum at $x = 1/n\sqrt{n}$.
(1 point)

(c) Recall that the sequence (f_n) converges uniformly to f on $[0, \infty)$ if and only if

$$\lim \left(\sup_{x \in [0,\infty)} |f_n(x) - f(x)| \right) = 0.$$

In our case we have that

$$\sup_{x \in [0,\infty)} |f_n(x) - f(x)| = |f_n(1/n\sqrt{n})| = \frac{\sqrt{n}}{2},$$

which is an unbounded sequence. Therefore, the sequence (f_n) does *not* converge uniformly to f on $[0, \infty)$.
(4 points)

(d) Note that $x \geq 2$ implies that $f_n'(x) < 0$ for all $n \in \mathbb{N}$. Hence,

$$\sup_{x \in [2,\infty)} |f_n(x) - f(x)| = |f_n(2)| = \frac{2n^2}{1+4n^3} < \frac{1}{2n} \to 0.$$

Therefore, the sequence (f_n) *does* converge uniformly to f on $[2, \infty)$.
(4 points)

Solution of Problem 6 (2 + 8 + 5 = 15 points)

(a) See figure.
(2 points)

(b) *Solution 1.* Setting $g(x) = f(x) - x$ gives

$$g(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ -1 & \text{if } 1 \le x \le 2. \end{cases}$$

Let $\epsilon > 0$ be arbitrary and consider the partition $P = \{0, 1 - \frac{1}{4}\epsilon, 1 + \frac{1}{4}\epsilon, 2\}$. Then

$$U(f, P) - L(f, P) = \tfrac{1}{2}\epsilon < \epsilon,$$

which shows that g is integrable on $[0, 2]$.
(4 points)

The function $h(x) = x$ is continuous and therefore integrable on $[0, 2]$.
(2 points)

Therefore, the sum $f(x) = h(x) + g(x)$ is integrable on $[0, 2]$.
(2 points)

Solution 2 (more work). We can also directly work with the function f itself, but we must be careful with choosing the partition since f has a discontinuity at $x = 1$. A convenient partition of the interval $[0, 2]$ is given by

$$x_k = \frac{k}{2n}, \quad k = 0, \ldots, 2n.$$

This is an equispaced partition of $[0, 2]$ in $2n$ subintervals, which means that each subinterval has length $x_k - x_{k-1} = 1/n$. Note that we have taken $2n$ subintervals, rather than n intervals. This particular choice ensures that $x_n = 1$, which makes it easier to handle the discontinuity.

Note that the function f is increasing on the intervals $[0, 1)$ and $[1, 2]$ and that we have a discontinuity at $x = 1$. As usual, we define

$$M_k = \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\} \quad \text{and} \quad m_k = \inf\{f(x) \,:\, x \in [x_{k-1}, x_k]\}.$$

Inspecting the graph of f implies that

$$M_k - m_k = \begin{cases} f(x_k) - f(x_{k-1}) & \text{if } k = 1, \ldots, n-1, \\ 1 & \text{if } k = n, \\ f(x_k) - f(x_{k-1}) & \text{if } k = n+1, \ldots, 2n. \end{cases}$$

(4 points)

Therefore, it follows that

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{k=1}^{2n} (M_k - m_k)(x_k - x_{k-1}) \\
&= \frac{1}{2n} \sum_{k=1}^{2n} (M_k - m_k) \\
&= \frac{1}{2n} \left(\sum_{k=1}^{n-1} [f(x_k) - f(x_{k-1})] + 1 + \sum_{k=n+1}^{2n} [f(x_k) - f(x_{k-1})] \right) \\
&= \frac{1}{2n} \Big(f(x_{n-1}) - f(x_0) + 1 + f(x_{2n}) - f(x_n) \Big) \\
&= \frac{1}{2n} \left(\frac{n-1}{2n} - 0 + 1 + 1 - 0 \right) = \frac{5n-1}{4n^2}.
\end{aligned}$$

(2 points)

Since $\lim(5n-1)/4n^2 = 0$ it follows that for every $\epsilon > 0$ we can take n large enough to guarantee that $U(f,P) - L(f,P) < \epsilon$, which implies that f is integrable on $[0,2]$.
(2 points)

(c) We have that

$$F(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \le x < 1, \\ \frac{1}{2}(x-1)^2 + \frac{1}{2} & \text{if } 1 \le x \le 2. \end{cases}$$

Hence, the difference quotient of F is given by

$$\frac{F(x) - F(1)}{x - 1} = \begin{cases} \frac{1}{2}(x+1) & \text{if } 0 \le x < 1, \\ \frac{1}{2}(x-1) & \text{if } 1 < x \le 2. \end{cases}$$

Now consider the sequences

$$x_n = 1 - \frac{1}{n} \quad \text{and} \quad y_n = 1 + \frac{1}{n}.$$

Then

$$\frac{F(x_n) - F(1)}{x_n - 1} \to 1 \quad \text{and} \quad \frac{F(y_n) - F(1)}{y_n - 1} \to 0,$$

which implies that F is *not* differentiable at $x = 1$. (Note: the Fundamental Theorem of Calculus cannot be applied since the function f is not continuous at $x = 1$.)
(5 points)